Int. J. Heat Mass Transfer. Vol. 33, No. 4, pp. 621–628, 1990
Printed in Great Britain

0017-9310/90 $3.00+0.00

Application of the hybrid method to inverse heat conduction problems

HAN-TAW CHEN and SHIUH-MING CHANG
Department of Mechanical Engineering, National Cheng Kung University, Tainan, Taiwan 70101, Republic of China

(*Received* 2 *June* 1989)

Abstract—The hybrid method involving the combined use of the Laplace transform method and the finite element method is considerably powerful for solving one-dimensional linear heat conduction problems. In the present method, the time-dependent terms are removed from the problem using the Laplace transform method and then the finite element method is applied to the space domain. The transformed temperature is inverted numerically to obtain the result in the physical quantity. The estimation of the surface heat flux or temperature from transient measured temperatures inside the solid agrees well with the analytical solution of the direct problem without Beck's sensitivity analysis and a least square criterion. Due to no time step, the present method can directly calculate the surface conditions of an inverse problem without step by step computation in the time domain until the specific time is reached. In addition, it is also not necessary to compute all the nodal temperatures at each time step when the present method is applied to an inverse problem. It is worth mentioning that a little effect of the measurement location on the estimates is shown in the present method. Thus, it can be concluded that the present method is straightforward and efficient for such problems.

INTRODUCTION

In recent years the analysis of inverse heat conduction problems has numerous applications in various branches of science and engineering, such as the prediction of the inner wall temperature of a reactor and the determination of the heat transfer coefficient and the outer surface conditions in the re-entry of a space vehicle. The difficulty in the analysis of inverse problems is due to the diffusive nature of heat flow. For this reason, slight inaccuracies in the measured interior temperatures will be magnified at the surface and may not be able to predict the surface conditions with the desired accuracy.

Various methods have been presented for the analysis of such problems. An exact solution of the linear inverse problem in conduction has been presented by Burggraf [1] under the condition of the instantaneous temperature for the given continuous temperature and heat flux histories at a given internal point. However, the results obtained by his method are also approximate for discrete or experimental data. Sparrow *et al.* [2] and Woo and Chow [3] applied the Laplace transform method to the present problem. Their solutions are valid only for small values of time (for large values of s), thus the application of their method is limited. The application of the finite element method to the inverse heat conduction problem has been investigated by Krutz *et al.* [4] and Busby and Trujillo [5]. The boundary element method in conjunction with Beck's sensitivity analysis has been presented for the solution of two-dimensional linear inverse heat conduction problems by Zabaras and Liu [6]. The finite difference analyses of the inverse problem have been introduced by a number of authors [7–9]. Mollification methods have been used by Murio [10, 11] to smooth the predictions at the surface. Recently, the method of regularizers was used to analyse some inverse problems [12–14]. Beck [15–17] used a least squares method to stabilize the inverse heat conduction problem. In all of the above methods, the complicated computational procedures are generally required in order to obtain a more accurate solution. For this reason, the present study presents a simple and efficient method which can obtain a stable and accurate solution without step by step computation in the time domain.

The present study is still limited to a one-dimensional planar geometry with constant thermal properties. The combined application of the Laplace transform method and the finite element method is applied to estimate surface conditions from data available at any location in a planar solid. This hybrid method has proved to be very powerful for solving linear transient heat conduction problems [18, 19]. This method is applied to analyse the present problem, some different test examples are illustrated. In these examples, a composite material is also included. It is found that the present method can improve the drawbacks of previous works such as the involved computation. The present technique may be applied to an inverse problem with temperature-dependent thermal conductivity. This will be studied in a future paper.

ANALYSIS

As shown in Fig. 1, one-dimensional linear heat conduction problems are selected as a basis for the

NOMENCLATURE

$[B^e]$	element gradient matrix	t	dimensionless time
c	specific heat of the material	v, w	free parameters
E	total number of elements	x	coordinate.
$\{f\}$	global force vector		
k	thermal conductivity of material	Greek symbols	
$[K]$	global conduction matrix	α	thermal diffusivity of material
l	distance between two nodes	α_{21}	ratio of thermal diffusivity, $k_2\rho_1c_1/k_1\rho_2c_2$
M	numbered node of the measurement location	γ	ratio of thermal conductivity, k_2/k_1
n	total number of nodes	θ	dimensionless temperature
$[N^e]$	element shape function matrix	$\tilde{\theta}$	transformed dimensionless temperature
N	sensor distance from heated surface	$\{\tilde{\theta}\}$	global transformed temperature vector
Q	heat generation inside the material	$\tilde{\theta}_M$	transformed dimensionless measured temperature
$q(t)$	surface heat flux		
s	Laplace transform parameter, $v+iw$	ρ	density of material.

application of the present method. The boundary at $x = 0$ is given as a uniform temperature or an insulated surface. The measured temperature at the location $x = N$ is assumed to be known as the function of time. It is desired to predict the heat flux and the temperature variation at $x = 1$. The dimensionless form for such problems may be expressed as

$$\frac{\partial^2\theta}{\partial x^2}+Q=\frac{\partial\theta}{\partial t} \quad \text{in } 0 \leqslant x \leqslant 1, t>0 \tag{1a}$$

$$\frac{\partial\theta(0,t)}{\partial x}=0 \quad \text{or } \theta(0,t)=0 \tag{1b}$$

$$\theta(N,t)=F(t) \tag{1c}$$

$$q(1,t)=\left.\frac{\partial\theta}{\partial x}\right|_{x=1} \quad \text{estimated} \tag{1d}$$

$$\theta(x,0)=0 \tag{1e}$$

where Q is the volume heat source and $\theta(N, t)$ the measured temperature at the location $x = N$. The unknown surface heat flux $q(1, t)$ and surface temperature $\theta(1, t)$ are to be estimated. In the present method, $q(1, t_s)$ at the specific time $t = t_s$ is guessed with Newton's iteration technique until the measured temperature $\theta(N, t_s)$ is satisfied. Note that the surface heat flux $q(1, t)$ and the surface temperature $\theta(1, t)$ are simultaneously determined without the need for Beck's sensitivity analysis and a least square criterion.

To remove the time-dependent terms from the differential equation and boundary conditions, the method of the Laplace transform will be employed. Taking the Laplace transform of equations (1) gives

$$\frac{d^2\tilde{\theta}}{dx^2}+\tilde{Q}=s\tilde{\theta} \quad \text{in } 0 \leqslant x \leqslant 1 \tag{2a}$$

subject to the transformed boundary conditions

$$\frac{d\tilde{\theta}(0)}{dx}=0 \quad \text{or } \tilde{\theta}(0)=0 \tag{2b}$$

$$\tilde{\theta}(N,s)=\tilde{F}(s) \tag{2c}$$

$$\tilde{q}=\left.\frac{d\tilde{\theta}}{dx}\right|_{x=1}. \tag{2d}$$

When a function $\phi(t)$ is given, its Laplace transform is defined as follows:

$$\tilde{\phi}(s)=\int_0^\infty \phi(t)\,e^{-st}\,dt. \tag{3}$$

The solution of equations (2) can be obtained by the Galerkin weighted residual process. The functional formulation that is equivalent to equation (2a) and its boundary conditions, equations (2b)–(2d), can be written as

$$I=\frac{1}{2}\int_0^1\left[\left(\frac{d\tilde{\theta}}{dx}\right)^2-2\tilde{Q}\tilde{\theta}+s\tilde{\theta}^2\right]dx-\tilde{q}\tilde{\theta}|_{x=1}. \tag{4}$$

Equation (4) can be written as the summation of individual elements. Assume that the temperature distribution in every individual element is linear [20]. Thus, the following elemental matrices can be given as

$$\tilde{\theta}^e=[N^e]\{\tilde{\theta}^e\} \tag{5a}$$

$$\{g^e\}=\left[\frac{d\tilde{\theta}}{dx}\right] \tag{5b}$$

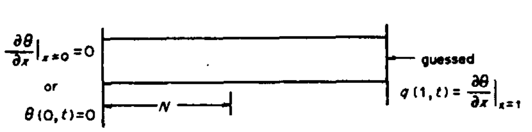

FIG. 1. Geometry of a slab.

$$[B^e] = \left[\frac{\partial N_1^e}{\partial x}\ \frac{\partial N_2^e}{\partial x}\right]. \tag{5c}$$

Equation (4) can be rewritten for each element as

$$I^e = \frac{1}{2}\int_0^l \{\tilde{\theta}^e\}^{\mathrm{T}}[B^e]^{\mathrm{T}}[B^e]\{\tilde{\theta}^e\}\,\mathrm{d}x - \int_0^l\left(\tilde{Q} - \frac{1}{2}s\{\tilde{\theta}^e\}^{\mathrm{T}}[N^e]^{\mathrm{T}}\right)[N^e]\{\tilde{\theta}^e\}\,\mathrm{d}x - \tilde{q}[N^e]\{\tilde{\theta}^e\}|_{x=l}. \tag{6}$$

The functional I can be expressed as

$$I = \sum_{e=1}^{E} I^e \tag{7}$$

where E denotes the total number of elements. The first derivative of the functional I with respect to $\{\tilde{\theta}\}$ must be equal to zero in order to minimize it, i.e.

$$\frac{\partial I}{\partial\{\tilde{\theta}\}} = \sum_{e=1}^{E}\frac{\partial I^e}{\partial\{\tilde{\theta}\}} = 0. \tag{8}$$

The rearrangement of equation (8) gives the following vector–matrix equation [18, 19]

$$[K]\{\tilde{\theta}\} = \{f\} \tag{9}$$

where $[K]$ is an $(n\times n)$ band matrix with a complex number, $\{\tilde{\theta}\}$ is an $(n\times 1)$ vector representing the unknown temperatures and $\{f\}$ is an $(n\times 1)$ vector representing the forcing terms.

It can obviously be found from equation (9) that the double direct Gaussian elimination algorithm and the numerical inversion of Laplace transforms [21–23] are applied to equation (9), then the solution located at a specific node in the given domain will be obtained. Under the circumstances, the location of the measured temperature can be considered as a specific node. It shows that if the guessed heat flux at $x = 1$ is substituted into equation (9), the calculated temperature at $x = N$ will be determined without computing all the nodal temperatures at each time step. Note that the estimation of the surface heat flux (or temperature) at a specific time is found without step by step computation in the time domain until the specific time is reached. Obviously, the numerical procedures of the present method are very different from those of previous works [4–17] which must compute all the nodal temperatures at each time step with step by step computation in the time domain. For this reason, the present hybrid method seems to be more efficient for such problems. The guessed heat flux at $x = 1$ can be determined by using Newton's iteration method when the relative error between the calculated temperature and the measured temperature at $x = N$ is less than a small number such as 0.0001. The expression of the double direct Gaussian elimination algorithm can be given as

$$\begin{bmatrix} \ddots \end{bmatrix}\begin{bmatrix}\tilde{\theta}_1\\ \cdot\\ \cdot\\ \tilde{\theta}_M\\ \cdot\\ \cdot\\ \tilde{\theta}_n\end{bmatrix} = \begin{bmatrix}q_1\\ \cdot\\ \cdot\\ q_M\\ \cdot\\ \cdot\\ q_n\end{bmatrix} \tag{10}$$

where $\tilde{\theta}_M = q_M$.

NUMERICAL EXAMPLES

To test the effectiveness of the present method some different cases are illustrated.

Example 1 (*one heat point-source*)

This example involves one heat point-source $Q = 1$ at the central position of a thick undeformable slab. Its initial temperature is equal to zero and the temperatures at the two ends of this slab are kept constant. Thus, the response of this system can be expressed as

$$0 \leqslant x \leqslant 1;\quad \frac{\partial^2\theta}{\partial x^2} + \delta(x-0.5) = \frac{\partial\theta}{\partial t} \tag{11}$$

subject to the conditions

$$\theta(0,t) = \theta(1,t) = 0 \tag{12a}$$

$$\theta(x,0) = 0. \tag{12b}$$

The exact solution to equations (11) and (12) is given as

$$\theta(x,t) = 2\sum_{m=1}^{\infty}\frac{1-\mathrm{e}^{-\beta_m^2 t}}{\beta_m^2}\sin\left(\frac{\beta_m}{2}\right)\sin(\beta_m x) \tag{13}$$

where $\beta_m = m\pi$, $m = 1, 2, 3, \ldots$

Assume that the boundary condition at $x = 1$ must be estimated, but the temperature at $x = N$ is measured. It implies that $\theta(N, t)$ is given as

$$\theta(N,t) = 2\sum_{m=1}^{\infty}\frac{1-\mathrm{e}^{-\beta_m^2 t}}{\beta_m^2}\sin\left(\frac{\beta_m}{2}\right)\sin(\beta_m N). \tag{14}$$

Under the circumstances, the system belongs to an inverse problem.

The Laplace transform of equation (11) with respect to time is

$$\frac{\mathrm{d}^2\tilde{\theta}}{\mathrm{d}x^2} + \frac{1}{s}\delta(x-0.5) = s\tilde{\theta}. \tag{15}$$

The resulting system of algebraic equations is given in the following matrix form when the finite element method is applied to Example 1

Table 1. Solutions for various measurement locations and nodes when $t = 0.6$

	$n = 9$		$n = 17$		
x	$M = 2$	$M = 7$	$M = 2$	$M = 15$	Exact
0.125	0.06231	0.06041	0.06237	0.05983	0.06230
0.250	0.12501	0.12119	0.12514	0.12001	0.12461
0.375	0.18850	0.18269	0.18870	0.18091	0.18701
0.500	0.25312	0.24524	0.25338	0.24283	0.24878
0.625	0.19418	0.18413	0.19454	0.18106	0.18701
0.750	0.13695	0.12459	0.13738	0.12082	0.12461
0.875	0.08159	0.06680	0.08212	0.06229	0.06230
1.000	0.02824	0.01086	0.02887	0.00557	0.00000
Heat flux at $x = 1$	−0.41863	−0.43994	−0.41779	−0.44636	−0.49813

n, total number of nodes; M, numbered node of the measurement location.

$$\begin{bmatrix} A & B & & & & \\ B & A & B & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & B & A & B \\ & & & & B & \frac{A}{2} \end{bmatrix} \cdot \begin{bmatrix} \tilde{\theta}_2 \\ \cdot \\ \cdot \\ \tilde{\theta}_c \\ \cdot \\ \cdot \\ \tilde{\theta}_n \end{bmatrix} = \begin{bmatrix} -B\tilde{\theta}_1 \\ 0 \\ \cdot \\ \cdot \\ \frac{1}{s} \\ 0 \\ \cdot \\ \cdot \\ \left.\frac{d\tilde{\theta}}{dx}\right|_{x=1} \end{bmatrix} \tag{16}$$

where $A = 2(1/l + sl/3)$ and $B = -1/l + sl/6$.

The surface heat flux (or temperature) is determined by using the hybrid method. Their calculated solutions are shown in Tables 1 and 2.

Example 2 (*heat flux and insulation at two ends*)

Consider a thick undeformable slab which is required to be heated by a surface heat flux $q(t)$ at $x = 1$. The surface at $x = 0$ is assumed to be insulated. Similarly, assume that its initial temperature is also zero. Using the 'data' generated in this manner, the surface heat flux $q(t)$ is reconstructed using the inverse technique. The response of this system is governed by

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial t}, \quad 0 \leqslant X \leqslant 1 \tag{17}$$

with the conditions

$$\frac{\partial\theta}{\partial x}(0, t) = 0 \tag{18a}$$

$$\frac{\partial\theta}{\partial x}(1, t) = q(t) = 1 \tag{18b}$$

$$\theta(x, 0) = 0. \tag{18c}$$

The exact solution of Example 2 is given as [24]

Table 2. Comparison of estimates with the exact solution at $t = 1$ and 5 when $n = 9$ and $M = 7$

	$t = 1$		$t = 5$	
x	Estimate	Exact	Estimate	Exact
0.125	0.06186	0.06250	0.06248	0.06250
0.250	0.12383	0.12499	0.12496	0.12499
0.375	0.18603	0.18750	0.18745	0.18751
0.500	0.24856	0.24931	0.24993	0.24932
0.625	0.18652	0.18750	0.18741	0.18751
0.750	0.12496	0.12499	0.12490	0.12499
0.875	0.06396	0.06250	0.06238	0.06250
1.000	0.00354	0.00000	—0.00014	0.00000
Heat flux at $x = 1$	−0.48099	−0.49981	−0.50014	−0.49984

Table 3. Solutions for various measurement locations and nodes when $t = 0.6$

	$n = 9$		$n = 17$		
x	$M = 2$	$M = 7$	$M = 2$	$M = 15$	Exact
0.000	0.43378	0.43322	0.43387	0.43370	0.43388
0.125	0.44157	0.44101	0.44166	0.44146	0.44165
0.250	0.46496	0.46437	0.46498	0.46479	0.46497
0.375	0.50398	0.50332	0.50390	0.50371	0.50385
0.500	0.55862	0.55790	0.55842	0.55821	0.55833
0.625	0.62894	0.62813	0.62857	0.62834	0.62844
0.750	0.71496	0.71404	0.71441	0.71416	0.71420
0.875	0.81669	0.81565	0.81592	0.81563	0.81564
1.000	0.93419	0.93299	0.93315	0.93282	0.93279
Heat flux at $x = 1$	1.00286	1.00157	1.00074	1.00036	1.00000

$$\theta(x,t) = t + \frac{3x^2-1}{6} - 2\sum_{m=1}^{\infty} (-1)^m \frac{\cos(\beta_m x)}{\beta_m^2} e^{-\beta_m^2 t} \tag{19}$$

where $\beta_m = m\pi$, $m = 1, 2, 3, \ldots$

To attempt to predict the surface conditions at $x = 1$ the temperature variation at the measurement location $x = N$ must be known such as Example 1. The present hybrid method is still applied to this test case. The calculated solutions are shown in Tables 3 and 4.

The calculated temperature distributions at various measurement locations for various nodes are listed in Tables 1 and 3, respectively. It is found from these two tables that the solutions of the 17 modelling nodes are in good agreement with those of the 9 modelling nodes. It implies that the present solutions are convergent. Tables 1 and 3 also show good agreement between the present solutions and the exact values even though the 9 modelling nodes are selected. Thus this modelling is used to evaluate the results under other conditions. It can be seen from Tables 1 and 3 that the changes of the measurement location have a little effect on the numerical solutions. Tables 2 and 4 show a comparison of the present solutions with the exact solutions when $t = 1$ and 5. Similarly, the good agreements are also observed in this comparison when the 9 modelling nodes are required. It is worth noting that the present solutions also agree well with the analytical solutions for the long-time cases. It can be concluded from these above facts that the present method is a powerful numerical technique for inverse heat conduction problems under more general conditions.

Example 3 (*composite materials with two layers*)

A composite slab of two layers with the assumptions of constant thermal properties and neglect of contact resistance is analysed. The dimensionless equations for the composite slab of two layers may be expressed as

$$0 \leqslant x \leqslant 0.5; \quad \frac{\partial^2 \theta_1}{\partial x^2} = \frac{\partial \theta_1}{\partial t} \tag{20a}$$

$$0.5 \leqslant x \leqslant 1; \quad \alpha_{21} \frac{\partial^2 \theta_2}{\partial x^2} = \frac{\partial \theta_2}{\partial t}. \tag{20b}$$

The boundary and initial conditions are given as follows:

Table 4. Comparison of estimates with the exact solution at $t = 1$ and 5 when $n = 9$ and $M = 2$

	$t = 1$		$t = 5$	
x	Estimate	Exact	Estimate	Exact
0.000	0.83329	0.83334	4.83324	4.83333
0.125	0.84113	0.84116	4.84131	4.84115
0.250	0.86459	0.86459	4.86459	4.86458
0.375	0.90371	0.90365	4.90375	4.90365
0.500	0.95847	0.95833	4.95845	4.95833
0.625	1.02889	1.02864	5.02893	5.02865
0.750	1.11496	1.11458	5.11481	5.11458
0.875	1.21667	1.21614	5.21646	5.21615
1.000	1.33401	1.33332	5.33344	5.33333
Heat flux at $x = 1$	1.00151	1.00000	1.00028	1.00000

$$t > 0, x = 0; \quad -\frac{\partial \theta_1(0,t)}{\partial x} = q(t) = |\cos 10t| \quad (21a)$$

$$x = 0.5; \quad \frac{\partial \theta_1}{\partial x} = \gamma \frac{\partial \theta_2}{\partial x}, \quad \theta_1 = \theta_2 \quad (21b)$$

$$x = 1; \quad \frac{\partial \theta_2}{\partial x} = 0 \quad (21c)$$

$$t = 0, 0 \leqslant x \leqslant 1; \quad \theta_1(x,0) = \theta_2(x,0) \quad (21d)$$

where

$$\alpha_{21} = \alpha_2/\alpha_1 = k_2\rho_1 c_1/k_1\rho_2 c_2,$$
$$\gamma = k_2/k_1 (\rho_1 c_1 = \rho_2 c_2 = 1, k_1 = 1, k_2 = 0.5).$$

Few papers have been published on heat transfer in the inverse problem of composite slabs [25]. Indeed, analytical results of heat transfer in such problems are often useful for many practical designs. Example 3 is applied to test the efficiency of the present method for the inverse heat conduction problem in the composite slab of two layers. In this test case the heat flux and temperature are assumed unknown at the location $x = 1$. Similarly, it is necessary to know the temperature variation at the measurement location $x = N$. The final vector–matrix equation to equations (20) and (21) using the Laplace transform method and the finite element method is given as

$$\begin{bmatrix} A_1 & B_1 & & & & & & \\ B_1 & 2A_1 & B_1 & & & & & \\ & \cdot & \cdot & \cdot & & & & \\ & & B_1 & A_1+A_2 & B_2 & & & \\ & & & B_2 & 2A_2 & B_2 & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \cdot & \cdot & \cdot \\ & & & & & & \cdot\ \cdot & \cdot \\ & & & & & & B_2 & A_2 \end{bmatrix} \cdot \begin{bmatrix} \bar{\theta}_1 \\ \bar{\theta}_2 \\ \cdot \\ \bar{\theta}_i \\ \cdot \\ \cdot \\ \bar{\theta}_M \\ \cdot \\ \bar{\theta}_n \end{bmatrix} = \begin{bmatrix} \bar{q}(s) \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad (22)$$

where 'i' and 'M' are numbered as the position of the interface and the location of the measurement, respectively. The values of A_1, B_1, A_2 and B_2 are given as

$$A_1 = \frac{1}{l} + \frac{sl}{3}, \quad B_1 = -\frac{1}{l} + \frac{sl}{6},$$

$$A_2 = \frac{\alpha_{21}}{l} + \frac{sl}{3}, \quad B_2 = -\frac{\alpha_{21}}{l} + \frac{sl}{6}. \quad (23)$$

Similar numerical procedures, as shown in Example 1, are used to analyse equation (22). The unknown surface heat flux $q(t)$ at a specific time is guessed and then checked whether the calculated temperature at the location $x = N$ matches the measured temperature using Newton's iteration technique. The numerical results are shown in Tables 5 and 6.

Table 5. Solutions for various measurement locations at $t = 0.11$ when $i = 9$ and $n = 17$

x	Inverse problem $M = 2$	Inverse problem $M = 12$	Direct problem
0.0000	0.26406	0.26176	0.26432
0.0625	0.23509	0.23385	0.23508
0.1250	0.20584	0.20506	0.20527
0.1875	0.17720	0.17617	0.17637
0.2500	0.15024	0.14917	0.14945
0.3125	0.12588	0.12486	0.12526
0.3750	0.10468	0.10367	0.10422
0.4375	0.08689	0.08589	0.08655
0.5000	0.07254	0.07154	0.07226
0.5625	0.05035	0.05005	0.05017
0.6250	0.03388	0.03318	0.03376
0.6875	0.02210	0.02203	0.02203
0.7500	0.01401	0.01336	0.01397
0.8150	0.00871	0.00861	0.00868
0.8750	0.00544	0.00543	0.00543
0.9375	0.00370	0.00368	0.00369
1.0000	0.00315	0.00313	0.00314
Heat flux at $x = 0$	0.45851	0.43799	0.45360

Table 5 shows solutions for the two different locations at $t = 0.11$ when $i = 9$ and $n = 17$. It is seen from Table 5 that no remarkable differences are observed in the comparisons for the two different measurement locations. It can be similarly concluded that the effect of the measurement location on the estimates is negligible. In addition, Tables 5 and 6 also show a fairly good agreement between the numerical solutions of the direct problem and the estimates.

CONCLUSIONS

An accurate and stable method of analysis is developed for solving one-dimensional linear inverse heat conduction problems including the composite slab of two layers. Due to no time step, the present method can directly predict the surface heat flux (or temperature) at a specific time from the transient measured temperature inside solids without step by step computation in the time domain until the specific time is reached. In other words, the solution to an inverse problem can be uniquely determined when the data at the interior location are known at a specific time. It is not necessary to compute all nodal temperatures at each time step when the present method is applied to the inverse problems. Furthermore, in the present model Beck's sensitivity or a least square criterion required in most previous works is not employed to best match the measurement. Thus, it can be concluded that the present method is more straight-

Table 6. Comparison between estimates and the results using the direct problem at $t = 0.06$ and 0.24 when $i = 9$, $M = 2$ and $n = 17$

	Inverse problem		Direct problem	
x	$t = 0.06$	$t = 0.24$	$t = 0.06$	$t = 0.24$
0.0000	0.25105	0.30978	0.25037	0.31127
0.0625	0.20143	0.26951	0.20142	0.26951
0.1250	0.15848	0.23522	0.15837	0.23537
0.1875	0.12209	0.20650	0.12171	0.20740
0.2500	0.09203	0.18273	0.09149	0.18440
0.3125	0.06795	0.16317	0.06740	0.16532
0.3750	0.04936	0.14707	0.04890	0.14933
0.4375	0.03566	0.13375	0.03532	0.13583
0.5000	0.02620	0.12264	0.02596	0.12436
0.5625	0.01428	0.10396	0.01418	0.10482
0.6250	0.00733	0.08799	0.00730	0.08814
0.6875	0.00353	0.07432	0.00353	0.07401
0.7500	0.00159	0.06289	0.00159	0.06234
0.8150	0.00067	0.05378	0.00067	0.05316
0.8750	0.00026	0.04714	0.00026	0.04653
0.9375	0.00010	0.04311	0.00010	0.04253
1.0000	0.00006	0.04176	0.00006	0.04119
Heat flux at $x = 0$				
	0.84756	0.72629	0.82534	0.73739

forward than previous works for such problems. It can also be found from the present study that the present method gives a little effect of the measurement location on the estimates. This implies that the present model offers a great deal of flexibility.

The present method is a considerably powerful numerical technique for linear one-dimensional heat conduction problems. Its mathematical formulation is so general that other geometries in addition to the planar one illustrated here can also be considered. A future paper may extend the present analysis to nonlinear inverse problems with temperature-dependent thermal properties.

Acknowledgement—The research reported here was performed under the auspices of the National Science Council, Taiwan.

REFERENCES

1. O. R. Burggraf, An exact solution of the inverse problem in heat conduction theory and applications, *ASME J. Heat Transfer* **86**, 373–382 (1964).
2. E. M. Sparrow, A. Haji-Sheikh and T. S. Lundgren, The inverse problem in transient heat conduction, *ASME J. Heat Transfer* **86**, 369–375 (1964).
3. K. C. Woo and L. C. Chow, Inverse heat conduction by direct inverse Laplace transform, *Numer. Heat Transfer* 4, 499–504 (1981).
4. G. W. Krutz, R. J. Schoenhals and P. S. Hore, Application of the finite element method to the inverse heat conduction problem, *Numer. Heat Transfer* **1**, 489–498 (1978).
5. H. R. Busby and D. M. Trujillo, Numerical solution to a two-dimensional inverse heat conduction problem, *Int. J. Numer. Meth. Engng* **21**, 349–359 (1985).
6. N. Zabaras and J. C. Liu, An analysis of two-dimensional linear inverse heat transfer problems using an integral method, *Numer. Heat Transfer* **13**, 527–533 (1988).
7. B. F. Blackwell, Efficient technique for the numerical solution of the one-dimensional inverse problem of heat conduction, *Numer. Heat Transfer* **4**, 229–238 (1981).
8. J. V. Beck, B. Litkouhi and C. R. Chair, Efficient sequential solution of the nonlinear inverse heat conduction problem, *Numer. Heat Transfer* **5**, 275–286 (1982).
9. E. Hensel and R. G. Hills, An initial value approach to the inverse heat conduction problem, *ASME J. Heat Transfer* **108**, 248–256 (1986).
10. D. A. Murio, The mollification method and the numerical solution of an inverse heat conduction problem. *SIAM J. Sci. Stat. Comput.* **2**(1), 17–34 (1981).
11. D. A. Murio, On the estimation of the boundary temperature or a sphere from measurements at its center, *J. Comp. Appl. Math.* **8**, 111–115 (1982).
12. M. R. Romanovsky, On regularization of inverse problems, *Teplofiz. Vysok. Temp.* **18**, 152–157 (1981).
13. S. Elubaev, Regularization of a particular inverse problem for the heat conduction equation, *Izv. Akad. Nauk. Kaz. SSR, Ser. Fiz. Mat. Nauk* No. 3, 21–26 (1980).
14. A. N. Tikhonov, V. B. Glasko and N. I. Kulik, On a regularizing algorithm for the solution of some inverse heat conduction problems, *Vestnik Mosk. Univ., Ser.* 3 *Fiz., Astronom.* **22**, 25–29 (1981).
15. J. V. Beck, Surface heat flux determination using an integral method, *Nucl. Engng Des.* **7**, 170–178 (1968).
16. J. V. Beck, Nonlinear estimation applied to the nonlinear inverse heat conduction problem, *Int. J. Heat Mass Transfer* **13**, 703–715 (1970).
17. M. Raynaud and J. V. Beck, Methodology for comparison of inverse heat conduction methods, *ASME J. Heat Transfer* **110**, 30–37 (1988).
18. H. T. Chen and C. K. Chen, Hybrid Laplace transform/finite element method for one-dimensional transient heat conduction problems, *Comput. Meth. Appl. Mech. Engng* **63**, 83–95 (1987).
19. H. T. Chen and C. K. Chen, Hybrid Laplace transform/finite element method for two-dimensional transient heat conduction, *J. Thermophys. Heat Transfer* **2**, 31–36 (1988).

20. L. J. Segerlind, *Applied Finite Element Analysis*. Wiley, New York (1976).
21. H. Dubner and J. Abate, Numerical inversion of Laplace transforms by relating them to the finite Fourier cosine transform, *J. ACM* **15**, 115–123 (1968).
22. F. Durbin, Numerical inversion of Laplace transform: efficient improvement to Dubner and Abate's method, *Comp. J.* **17**, 371–376 (1973).
23. G. Honig and U. Hirdes, A method for the numerical inversion of Laplace transforms, *J. Comp. Appl. Math.* **9**, 113–132 (1984).
24. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (2nd Edn). Oxford University Press, Oxford (1959).
25. R. G. Hills, G. P. Mulholland and L. K. Mathews, The application of the Backus–Gilbert method to the inverse heat conduction problem in composite media, ASME Paper No. 82-HT-26 (1982).

APPLICATION DE LA METHODE HYBRIDE AU PROBLEME DE CONDUCTION THERMIQUE INVERSE

Résumé—La méthode hybride, utilisant la méthode de la transformation de Laplace et la méthode des éléments finis, est particulièrement puissante pour résoudre le problème linéaire monodimensionnel de la conduction. Dans la présente méthode, les termes dépendants du temps sont écartés en utilisant la méthode de Laplace et ensuite la méthode des éléments finis est appliquée au domaine spatial. La température transformée est inversée numériquement pour obtenir les résultats dans le domaine réel physique. L'estimation des flux et des températures de surface, à partir des températures mesurées à l'intérieur du solide, s'accorde bien avec la solution analytique du problème direct sans l'analyse de sensibilité de Beck et le critère des moindres carrés. La présente méthode peut calculer directement les conditions de surface d'un problème inverse sans le calcul pas à pas dans le domaine jusqu'à ce que le temps spécifique soit atteint. De plus, il n'est pas nécessaire de calculer toutes les températures nodales à chaque pas de temps lorsque la méthode est appliquée au problème inverse. On peut préciser qu'on constate un effet réduit du lieu de mesure sur l'estimation. On estime qu'une telle méthode est simple et efficace pour traiter de tels problèmes.

ANWENDUNG DER HYBRID-METHODE AUF INVERSE WÄRMELEITPROBLEME

Zusammenfassung—Die Hybrid-Methode, welche eine kombinierte Nutzung der Laplace-Transformation und der Finite-Elemente-Methode beinhaltet, ist für die Lösung eindimensionaler linearer Wärmeleitprobleme gut geeignet. Bei der vorgestellten Methode werden die Terme, die die Zeit enthalten, durch Anwendung der Laplace-Transformation aus dem Problem entfernt. Danach wird im Raumbereich die Finite-Elemente-Methode angewandt. Die numerische Rücktransformation der Temperatur liefert deren Werte als physikalische Größen. Die Bestimmung von Oberflächenwärmestrom und -temperatur aus den transient gemessenen Temperaturen im Körper stimmt ohne Empfindlichkeitsanalyse nach Beck und die Methode der kleinsten Fehlerquadrate gut mit der analytischen Lösung des direkten Problems überein. Da Zeitschritte entfallen, kann die vorgestellte Methode die Oberflächenbedingungen eines inversen Problems direkt in der Zeitebene bestimmen. Außerdem ist es auch nicht notwendig, alle Knotentemperaturen jedes Zeitschritts zu berechnen, wenn die vorgestellte Methode auf ein inverses Problem angewandt wird. Es ist erwähnenswert, daß sich durch den Einfluß der Meßposition auf das Ergebnis ein kleiner Effekt zeigt. Es kann gefolgert werden, daß die vorgestellte Methode auf solche Probleme erfolgreich angewandt werden kann.

ПРИМЕНЕНИЕ КОМБИНИРОВАННОГО МЕТОДА К ОБРАТНЫМ ЗАДАЧАМ ТЕПЛОПРОВОДНОСТИ

Аннотация—Для решения одномерных задач теплопроводности весьма эффективным является метод на основе совместного использования преобразования Лапласа и метода конечных элементов. Согласно этому методу зависящие от времени слагаемые исключаются из задачи при помощи преобразований Лапласа, а затем к пространственной области применяется метод конечных элементов. Преобразованная температура обращается численно для получения физических результатов. Оценка теплового потока или температуры на поверхности по измеренным переходным температурам в твердом теле хорошо согласуется с аналитическим решением прямой задачи без анализа чувствительности Бека и критериев наименьших квадратов. В силу отсутствия временного скачка данный метод позволяет непосредственно рассчитать условия на поверхности в обратной задаче без последовательного расчета во временной области до достижения конкретного момента времени. Кроме того, при использовании данного метода к обратной задаче отпадает необходимость определения всех температур в точках сетки на каждом временном шаге. Следует отметить, что при использовании данного метода выбор места измерений оказывает небольшое влияние на оценки. Таким образом, данный метод является прямым и эффективным при решении таких задач.